

# Universal trapping scaling on the unstable manifold for a collisionless electrostatic mode

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## Abstract

An amplitude equation for an unstable mode in a collisionless plasma is derived from the dynamics on the two-dimensional unstable manifold of the equilibrium. The mode amplitude  $\rho(t)$  decouples from the phase due to the spatial homogeneity of the equilibrium, and the resulting one-dimensional dynamics is analyzed using an expansion in  $\rho$ . As the linear growth rate  $\gamma$  vanishes, the expansion coefficients diverge; a rescaling  $\rho(t) \equiv \gamma^2 r(\gamma t)$  of the mode amplitude absorbs these singularities and reveals that the mode electric field exhibits trapping scaling  $|E_1| \sim \gamma^2$  as  $\gamma \rightarrow 0$ . The dynamics for  $r(\tau)$  depends only on the phase  $e^{i\xi}$  where  $d\epsilon_k/dz = |\epsilon_k|e^{-i\xi/2}$  is the derivative of the dielectric as  $\gamma \rightarrow 0$ .

The collisionless evolution of an unstable electrostatic mode is a fundamental topic in the theory of strong wave-particle interactions. The linear instability arises from a resonant interaction between an initial electrostatic fluctuation and particles at the phase velocity of the linear mode; the nonlinear evolution is marked by the trapping of the resonant particles in the wave potential and decaying oscillations in the wave amplitude due to the bouncing and phase mixing of the trapped particles. The difficulty of treating the dynamics of this process analytically is well known, and the extensive literature tends to focus on certain special regimes where simplifying approximations are possible; for example, instabilities due to a cold low density beam [1] or a gentle “bump on tail” [2]- [9].

Early work on the interaction of a narrow spectrum of weakly unstable waves with a cold electron beam identified the importance of particle trapping and predicted a relation  $\omega_b \sim \gamma$  between the linear growth rate  $\gamma$  and the bounce frequency  $\omega_b^2 = ekE_k/m$  in the nonlinear state that emerges after the linear instability has saturated. [1] This relation is more transparently stated as a property of the electric field  $E_k$  of the saturated wave:  $E_k \sim \gamma^2$  as  $\gamma \rightarrow 0$ ; a property we refer to as trapping scaling for  $E_k$ . Initial studies of the saturated state for the bump on tail instability also predicted trapping scaling. [2,3] These investigations all invoke approximations treating the response of the non-resonant electrons as linear or adiabatic.

By contrast, Simon and Rosenbluth constructed a time-asymptotic state by perturbatively expanding the Vlasov equation in the beam density and demanding that secular terms vanish at each order. Their procedure led to expressions involving singular functions which were defined by prescribing certain regularization procedures; the resulting theory predicted nonlinear states with much larger electric fields  $E_k \sim \sqrt{\gamma}$ . [4] Subsequent perturbative calculations by other groups involve similar assumptions and reach the same basic conclusion [5,6] with one exception: Larsen has studied the bump on tail problem using a multiple scale expansion in time and velocity; in his formulation, a “singular layer” at the phase velocity leads him to posit trapping scaling as an ansatz. [7] Numerical simulations of the instability find the initial, possibly metastable, nonlinear state with trapping scaling [8,9] but have

sometimes claimed to detect a slow growth in  $E_k$  on very long time scales. [9] Recent laboratory experiments of an electron beam interacting with an electrostatic wave (supported by a travelling wavelube) find the nonlinearly saturated wave amplitude is described by trapping scaling over the length scale of the experiment. [10]

In this paper we describe a new approach which also treats the Vlasov equation perturbatively, but simplifies the problem in a new way: the initial conditions are restricted so that the evolution occurs on the unstable manifold of the equilibrium  $F_0$ . [11] Physically this restriction means that only the unstable mode is initially excited, rather than viewing the mode as one component of an arbitrary fluctuation. Mathematically the unstable manifold provides a finite-dimensional setting which partially compensates for the absence of a finite-dimensional center manifold in this problem. In the simplest case, the unstable manifold defines a two-dimensional problem where the invariance of  $F_0$  under spatial translation implies that the evolution of the mode amplitude decouples from the phase and is described by a one-dimensional dynamical system. We formulate and analyze this one-dimensional problem.

The dimensionless equations for the potential  $\phi(x, t)$  and electron distribution function  $F(x, v, t)$  are

$$\partial_t F + v \partial_x F + \partial_x \phi \partial_v F = 0 \quad \partial_x^2 \phi = \int_{-\infty}^{\infty} dv F - 1 \quad (1)$$

where

$$\int_{-L/2}^{L/2} dx \int_{-\infty}^{\infty} dv F(x, v, t) = 1. \quad (2)$$

Given an equilibrium  $F_0(v, \mu)$ , Eq.(1) gives the evolution equation for  $f(x, v, t) \equiv F(x, v, t) - F_0(v, \mu)$ ,

$$\partial_t f = \mathcal{L}f + \mathcal{N}(f) \quad (3)$$

where  $\mathcal{N}(f) = -\partial_x \phi \partial_v f$  and  $\mathcal{L}f = -v \partial_x f - \partial_x \phi \partial_v F_0$ ; here  $\mu$  denotes parameters such as density or temperature that determine the properties of  $F_0$ . We assume periodic boundary conditions on  $f$  thus the Fourier components  $f_k(v) e^{ik k_c x}$  are discrete multiples of  $k_c = 2\pi/L$ .

The length of the system is chosen so that as  $\mu$  varies an instability occurs for  $k_c$  (or  $k = 1$ ) corresponding to an eigenfunction  $\mathcal{L}\Psi = \lambda\Psi$  where  $\Psi(x, v) = e^{ik_c x} \psi_c(v)$ ,  $\lambda = -ik_c z_0$ , and  $\psi_c(v) = \partial_v F_0/k_c^2(v - z_0)$ . The unstable mode is determined by a root  $z_0 = v_p + i\gamma/k_c$  of the dielectric function

$$\epsilon_k(z) \equiv 1 - \frac{1}{(kk_c)^2} \int_{-\infty}^{\infty} dv \frac{\partial_v F_0(v, \mu)}{v - z} \quad (\text{Im } z > 0) \quad (4)$$

and both  $v_p$ , the phase velocity of the mode, and  $\gamma$  depend on the equilibrium parameters  $\mu$ . For our purposes it is not important exactly what parameters  $\mu$  represents or how they are varied, the mode is assumed to be neutrally stable for  $\mu = \mu_c$  and to become unstable if  $\mu$  is appropriately shifted away from  $\mu_c$ ; the inverse limit  $\mu \rightarrow \mu_c$  from the unstable regime will be denoted by  $\gamma \rightarrow 0^+$ . In addition we assume the derivatives  $d^n \epsilon_k(z_0)/dz^n$  have finite limits as  $\gamma \rightarrow 0^+$  and that the first derivative  $\epsilon'_k(z_0)$  is non-zero. This latter requirement that  $z_0$  is a simple root will typically be satisfied when a single parameter is varied, e.g. beam density or beam velocity.

When  $F_0(v, \mu)$  lacks reflection symmetry as in a beam-plasma system then  $v_p \neq 0$  and the eigenvalue  $\lambda$  is complex. For  $\gamma > 0$  there is a complex conjugate pair of eigenvalues  $(\lambda, \lambda^*)$  in the right half plane and a symmetrically placed pair  $(-\lambda, -\lambda^*)$  in the left half plane; as  $\gamma \rightarrow 0^+$  this eigenvalue quadruplet merges into the continuous spectrum on the imaginary axis. [12] For reflection-symmetric problems,  $F_0(v, \mu) = F_0(-v, \mu)$ , the eigenvalue can be real or complex; for example, two-stream instabilities correspond to real eigenvalues of multiplicity two:  $\Psi$  and  $\Psi^*$  are linearly independent eigenvectors for  $\lambda$ . [13] An instability with complex  $\lambda$  leads to four-dimensional unstable manifolds when  $F_0$  is reflection-symmetric and will not be considered here. [14]

For  $\lambda$  complex (no reflection symmetry) or  $\lambda$  real (with reflection symmetry), we decompose  $f$  to isolate the two unstable modes

$$f(x, v, t) = [A(t)\Psi(x, v) + cc] + S(x, v, t) \quad (5)$$

where  $A(t) = (\tilde{\Psi}, f)$  and  $(\tilde{\Psi}, S) = 0$ ;  $\tilde{\Psi}$  is the adjoint eigenvector corresponding to  $\lambda^*$  and

$(G_1, G_2) \equiv \int dx \int dv G_1(x, v)^* G_2(x, v)$  denotes the inner product. The equations for  $A$  and  $S$  follow from (3)

$$\dot{A} = \lambda A + (\tilde{\Psi}, \mathcal{N}(f)) \quad (6)$$

$$\partial_t S = \mathcal{L}S + \mathcal{N}(f) - [(\tilde{\Psi}, \mathcal{N}(f)) \Psi + cc]; \quad (7)$$

the linear terms are now decoupled, but nonlinear couplings between  $\dot{A}$  and  $\partial_t S$  remain. For  $\gamma > 0$ , the modes  $\Psi$  and  $\Psi^*$  span the two-dimensional unstable subspace  $E^u$  which is invariant under the linear flow  $\partial_t f = \mathcal{L}f$ . The nonlinear terms couple the unstable modes to the two stable modes (spanning the stable subspace  $E^s$ ) and to the continuum (which spans the infinite-dimensional center subspace  $E^c$ ); these interactions bend the unstable subspace into a two-dimensional unstable manifold which is invariant for the full nonlinear evolution. Solutions on this manifold asymptotically approach  $F_0$  at an exponential rate  $e^{\gamma t}$  as  $t \rightarrow -\infty$ .

At the equilibrium, the manifold is tangent to  $E^u$  and thus can be represented as the graph of a function  $H : E^u \rightarrow E^c \oplus E^s$ ; see Figure 1. With respect to the decomposition in (5), when  $f$  is a point on the unstable manifold (denoted  $f^u$ ) then

$$f^u(x, v) = [A\Psi(x, v) + cc] + H(x, v, A, A^*); \quad (8)$$

hence the time dependence of  $S$  for a solution on the manifold is determined by the dynamics of  $A(t)$  and the geometry of the manifold:  $S(x, v, t) = H(x, v, A(t), A^*(t))$ . For such a solution  $\partial_t S = \dot{A} \partial_A H + \dot{A}^* \partial_{A^*} H$  and consistency with (7) requires

$$\left[ \dot{A} \partial_A H + \dot{A}^* \partial_{A^*} H \right] \Big|_{f=f^u} = \mathcal{L}H + \mathcal{N}(f^u) - [(\tilde{\Psi}, \mathcal{N}(f^u)) \Psi + cc]. \quad (9)$$

Solving this equation for  $H$ , then determines the representation of  $f^u$  in (8). Setting  $f = f^u$  in (6) yields the desired description of the dynamics on the unstable manifold:

$$\dot{A} = \lambda A + (\tilde{\Psi}, \mathcal{N}(f^u)) \equiv V(A, A^*, \mu); \quad (10)$$

this is an autonomous two-dimensional dynamical system for  $A(t)$ . For small  $A$ , the graph function  $H$  is second order in  $A$  and  $f^u \approx [A(t)\Psi(x, v) + cc]$ , in this sense the dynamics on the unstable manifold corresponds to an initial excitation of only the unstable modes.

The spatial translation symmetry of the problem constrains  $V$  to have the form  $V(A, A^*, \mu) = Ap(|A|^2, \mu)$  where  $p$  is an undetermined function. If  $F_0$  is reflection-symmetric as in the two-stream instability, then  $p$  must be real-valued; otherwise  $p$  is complex-valued. Finally in amplitude/phase notation  $A = \rho e^{-i\theta}$  (10) becomes

$$\dot{\rho} = \rho \operatorname{Re}[p(\rho^2, \mu)] \quad \rho \dot{\theta} = -\operatorname{Im}[p(\rho^2, \mu)]; \quad (11)$$

thus on the unstable manifold the evolution of the mode amplitude  $\rho(t)$  decouples from the phase  $\theta(t)$  and is a one-dimensional problem.

For small wave amplitudes, we have investigated the properties of (11) by representing  $p(\rho^2, \mu)$  as a power series,

$$p(\rho^2, \mu) = \sum_{j=0}^{\infty} p_j(\mu) \rho^{2j}. \quad (12)$$

Clearly  $p_0(\mu) = \lambda$  from (6) and the higher order coefficients  $p_j(\mu)$  are determined by solving (9) for  $H$  as a power series in  $(A, A^*)$  then substituting into (10); this calculation will be presented elsewhere. [15] Of greatest interest are the properties of  $p_j$  in the regime of weak instability  $\gamma \rightarrow 0^+$ : for  $j > 0$ , these coefficients are singular at every order

$$p_j(\mu) = \frac{b_j(\mu_c)}{\gamma^{4j-1}} [1 + \mathcal{O}(\gamma)], \quad (13)$$

and the remaining asymptotic dependence on the equilibrium  $F_0$  is remarkably simple. Each  $b_j(\mu_c)$  depends on  $F_0$  only through the phase  $e^{i\xi(\mu_c)} \equiv \epsilon'_k(v_p)^*/\epsilon'_k(v_p)$ , which is determined by the limiting value of  $\epsilon'_k(z_0)$ . More precisely, at each order there is a calculable function  $Q_j$ , independent of  $F_0$ , such that

$$b_j(\mu_c) = Q_j(e^{i\xi(\mu_c)}); \quad (14)$$

in particular, we find  $b_1(\mu_c) = -1/4$  although in general  $Q_j$  will not be constant at higher order.

The significance of the divergence in (13) is clearer when we introduce a rescaled mode amplitude  $\rho(t) \equiv \gamma^2 r(\gamma t)$  which varies on the slow time scale  $\tau \equiv \gamma t$  and rewrite the dynamics (11) as

$$\frac{dr}{d\tau} = r \left\{ 1 + \sum_{j=1}^{\infty} \text{Re} [b_j(\mu_c) + \mathcal{O}(\gamma)] r^{2j} \right\} \quad \dot{\theta} = \omega - \gamma \sum_{j=1}^{\infty} \text{Im} [b_j(\mu_c) + \mathcal{O}(\gamma)] r^{2j} \quad (15)$$

where  $\omega = k_c v_p$  is the mode frequency. The  $\gamma \rightarrow 0^+$  limit for (15) is nonsingular and the asymptotic equations have some notable features. First, the dependence on  $F_0$  is entirely contained in the three parameters  $\gamma, \omega$ , and  $e^{i\xi}$ ; in particular at  $\gamma = 0$  the rescaled amplitude dynamics depends only on  $e^{i\xi}$ . If this phase is fixed, then any variations in densities or temperatures characterizing  $F_0$  do not affect the evolution of  $r(\tau)$ . For example, a beam-plasma instability (complex  $\lambda$ ) and a two-stream instability (real  $\lambda$ ), compared at a fixed value of  $e^{i\xi}$ , have identical amplitude equations up to  $\mathcal{O}(\gamma)$  corrections. A second feature is that the linear term defines a growth rate that is not small, i.e. not  $\mathcal{O}(\gamma)$ , rather the growth rate is unity. Moreover, unless  $b_j$  happens to vanish, *all* the higher order terms  $r^{2j}$  in the amplitude equation are order unity in the limit  $\gamma \rightarrow 0^+$ . Thus there is no small parameter in (15) to justify a truncation of the series, and consequently it is not straightforward to calculate the time-asymptotic amplitude  $r(\tau \rightarrow \infty)$ . However, assuming that  $r(\tau)$  approaches a limiting value  $r_\infty$  as  $\tau \rightarrow \infty$ , then it follows that this time-asymptotic state is a BGK mode. [15]

The implications of (13) for the behavior of the electric field  $|E_1|$  at  $k_c$  (or  $k = 1$ ) follow from  $ik_c E_1 = \int dv f_1(v, t) = A(t) + \int dv H_1(v, A, A^*)$ ; one can show that

$$k_c |E_1(t)| = \gamma^2 r [1 + r^2 \hat{\Gamma}_1(r^2, e^{i\xi}) + \mathcal{O}(\gamma)] \quad (16)$$

where  $\hat{\Gamma}_1$  represents a power series in  $r^2$  whose coefficients depend on  $F_0$  only through  $e^{i\xi}$ . [15] This expression predicts that the trapping scaling  $|E_1| \sim \gamma^2$  is a universal characteristic of the entire evolution; in particular it should hold for the time-asymptotic state. This conclusion agrees with the early work on the instability due to a small cold beam, [1] and generalizes it to a much wider class of instabilities.

An interesting aspect of (15) - (16) is the apparent absence of the familiar trapping oscillations in the evolution of  $|E_1|$ . This evolution is determined by  $r(\tau)$  whose dynamics is described by an autonomous one-dimensional flow, and it is well known that smooth one-

dimensional equations cannot describe oscillations. We have conjectured elsewhere [16], on the basis of simpler exactly solvable models, that the unstable manifold develops a spiral structure away from the equilibrium as illustrated in Figure 2. If this is correct, then representing the dynamics on the manifold via a mapping  $H$  from the unstable subspace yields a vector field on  $E^u$  with branch point singularities at the points where the flow moves from one branch of the spiral to the next. Thus  $\text{Re}[p(\rho^2, \mu)]$  would have a branch point  $\text{Re}[p(\rho^2, \mu)] \sim \sqrt{\rho_b^2 - \rho^2}$  at  $\rho_b^2$  and as the mode grows  $\rho^2(\tau) \rightarrow \rho_b^2$  would signal the onset of trapping oscillations with the passage of the trajectory to the next branch of the unstable manifold. Note that a trajectory will reach such a node in finite time, unlike the more familiar situation of a node where the vector field is differentiable and the approach time is infinite. In addition the loss of smoothness at  $\rho = \rho_b$  introduces the lack of uniqueness needed by the solution to pass through the branch point. Such a spiral structure would present a significant obstacle to using the power series (12) to determine the time-asymptotic amplitude  $r_\infty$ .

The coefficients  $p_j(\mu)$  are calculated as integrals over velocity and the singular behavior in (13) arises from pinching singularities that develop at  $v = v_p$  as  $\gamma \rightarrow 0^+$ . The regularization procedures proposed in previous treatments would modify these integrals to remove the pinching singularity and eliminate the divergences reflected in (13). Any such regularization replaces trapping scaling by  $|E_1| \sim \sqrt{\gamma}$  which is typical of a Hopf or pitchfork bifurcation in which there is no continuous spectrum on the imaginary axis. In these latter bifurcations the coefficients  $p_j$  are nonsingular, the series for  $p$  can be truncated, and time-asymptotic state found by balancing the linear term against a cubic nonlinearity. However, our calculations show that the singularities in  $p_j$  found here simply imply a different dependence of  $\rho(t)$  and  $|E_1|$  on  $\gamma$  as  $\gamma \rightarrow 0^+$ , and that there is no need to introduce an ad hoc regularization.

Instabilities in other Hamiltonian systems, including ideal shear flows [17] and solitary waves [18], also exhibit key features of this problem, most notably that the unstable modes correspond to eigenvalues emerging from a neutral continuum at onset. It will be interesting to determine if similar singularities arise in the amplitude equations for the unstable modes in these problems. [19] By contrast, there is at least one example, a phase model for the onset of



synchronized behavior in a population of oscillators, in which the critical eigenvalues emerge from the continuum at the onset of instability but the amplitude equations are nonsingular and  $\sqrt{\gamma}$  scaling is found (at least in the best understood case of a real eigenvalue). [20,21] This difference in the nonlinear behavior seems noteworthy since the linear dynamics of the model is qualitatively similar to Vlasov although apparently lacking a Hamiltonian structure. [22]

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## FIGURES

FIG. 1. Local geometry of the unstable manifold; the equilibrium  $F_0$  is at the origin

FIG. 2. Conjectured spiral structure in the global unstable manifold